Dynamic direct access of MSO query evaluation over strings

Pierre Bourhis, *Florent Capelli*, Stefan Mengel and Cristian Riveros
CRIL, Université d’Artois
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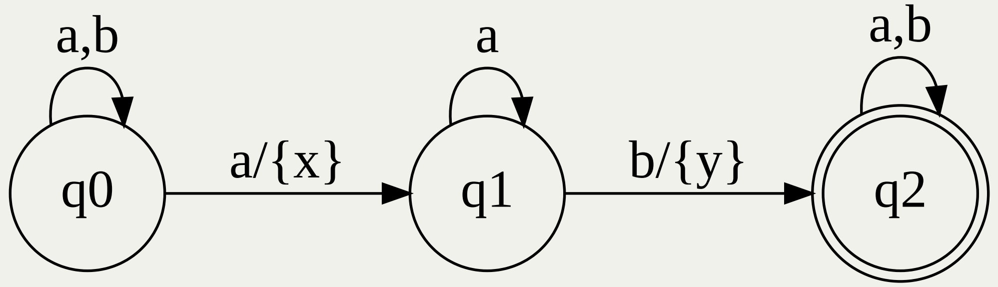
Variable Set Automata

Tagging positions in words



w	b	a	a	b	a	a	a	b	a	b	a	a
Positions	1	2	3	4	5	6	7	8	9	10	11	12

$[\![A]\!](w)$	x	y

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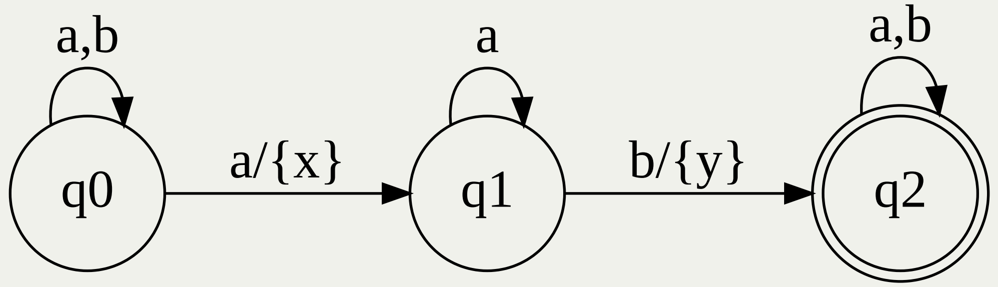

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(run 1)	2	4

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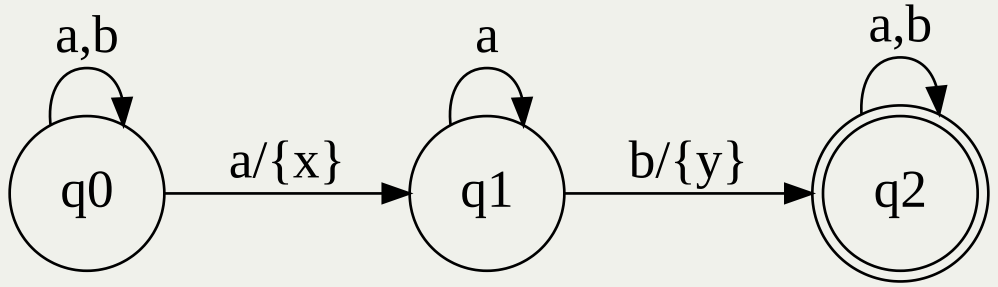

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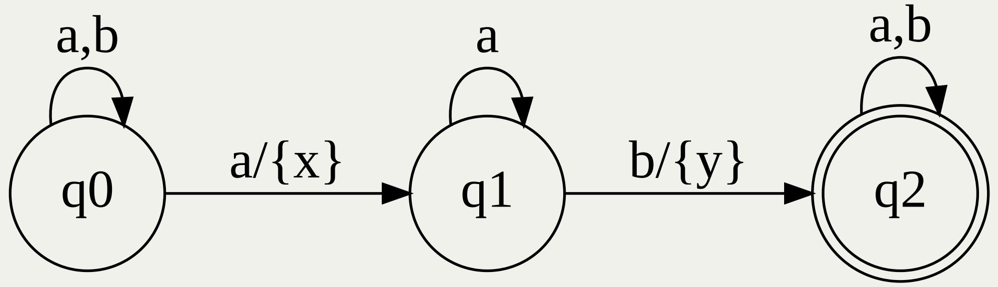

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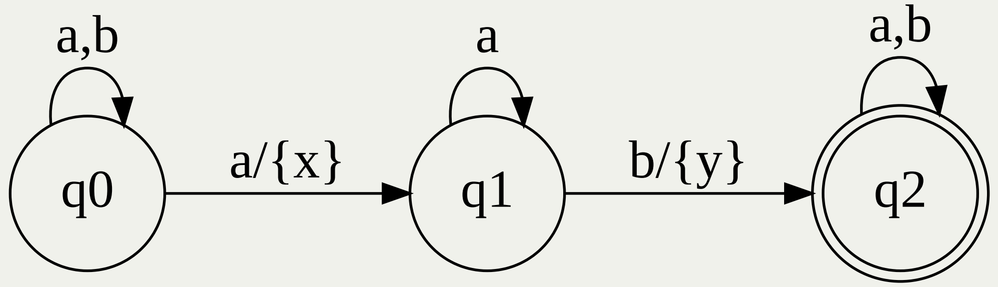

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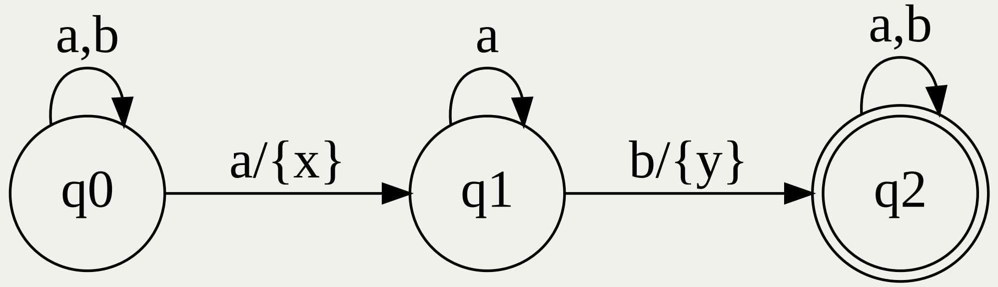

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Build a data structure allowing to access each tuple in $[\![A]\!](w)$ efficiently.

Related formalisms

VSet Automata are akin to:

- **Document spanners**: “span” can be encoded as two variables s_x (start x) and e_x (end x)

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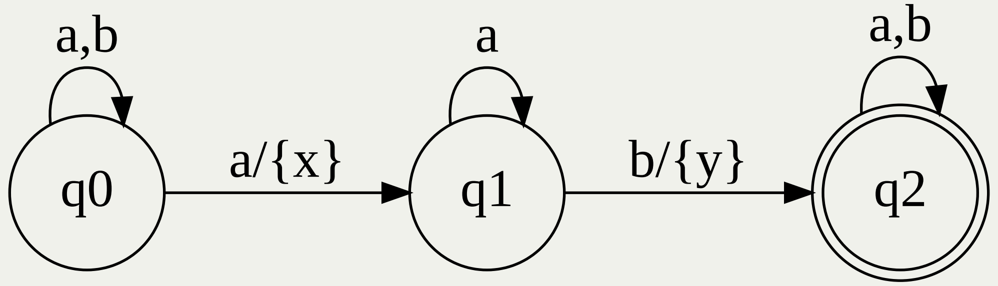
VSet Automata are akin to:

- **Document spanners**: “span” can be encoded as two variables s_x (start x) and e_x (end x)
- **MSO over words**:
    - relation $a(x)$ for each letter a: “there is an a at position x
    - order $<$ on positions
    - first order and monadic second order quantifications
    - **Theorem**: for each such formula φ, there exists a vset automate A_φ such that $[\![A_\varphi]\!] = [\![\varphi]\!]$.

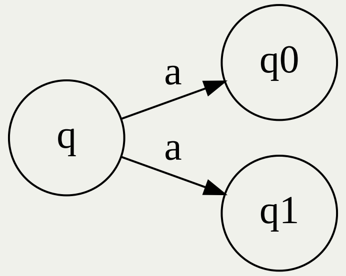
$$\varphi(x,y) \equiv a(x) \wedge b(y) \wedge \forall z((x < z \wedge z < y) \Rightarrow a(z))$$



vset automata A_φ for φ

Desirable properties

- **Determinism**: two edges going out of state q have distinct labels.



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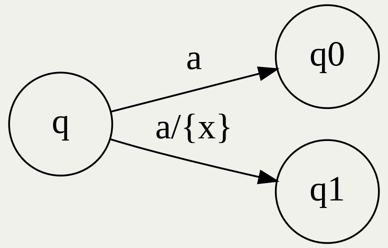

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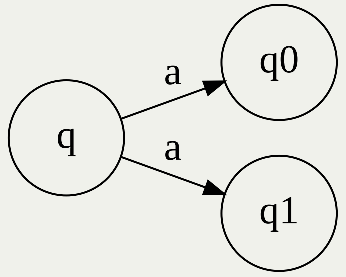

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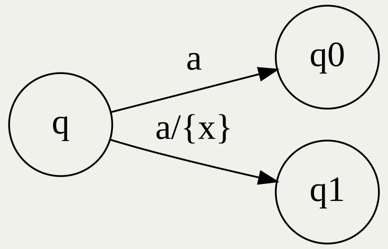

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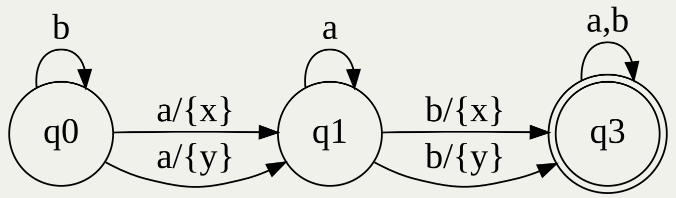
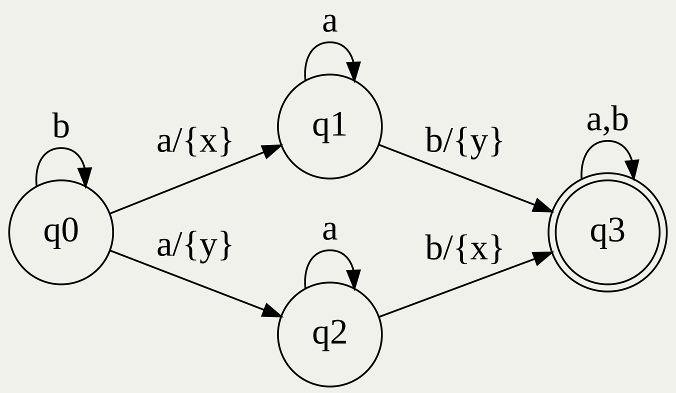
- **Functionality**: every path from q_0 to a final state tags each variable exactly once.



Not functional



Functional version

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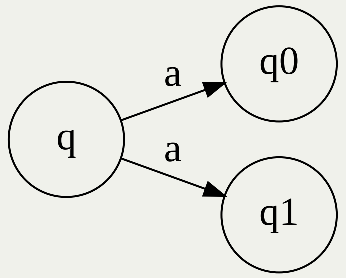

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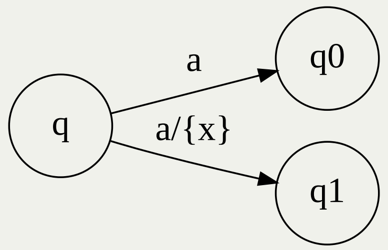

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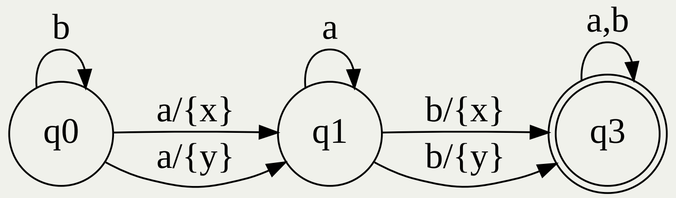

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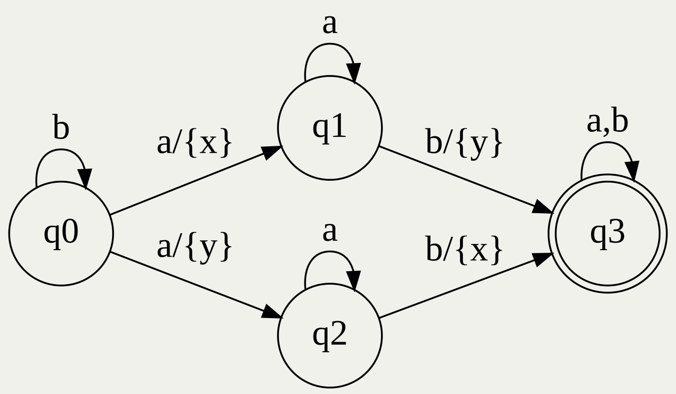

Functional version

Functionnality: every q is mapped with X_q, variables tagged before reaching q.

Normalization

Let A be an vset automaton. One can construct a deterministic function vset automaton A' such that $[\![A]\!] = [\![A']\!]$.

- Intuition: automata over states $2^{Q \cup X}$.
- A' may be of size $exp(A)$

In this talk: **data complexity model** where w is the *data* and A is the *query*.

$\Rightarrow$ A is considered constant, hence assumed to be deterministic and functional.

Direct Access for vset automata

Direct access queries

Fix a (deterministic functional) automaton $A(x_1, \ldots, x_k)$ (considered constant).

A **direct access query** for a word $w \in \Sigma^*$:

- input integer k,
- output the kth tuple of $[\![A]\!](w)$ or,
- **fails** if $k > \#[\![A]\!](w)$.

where $[\![A]\!](w)$ is ordered by lexicographical ordering on $[n]^X$.

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Direct access complexity

Given $w \in \Sigma^*$ of length n:

- **Precomputation phase**: construct a data structure D_w in time $p(n)$.
- **Access phase**: given k, output $[\![A]\!](w)[k]$ in time $a(n)$ using D_w.

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Can we have better preprocessing without hurting access time too much?

Contribution

We show that we can solve direct access for $[\![A]\!](w)$ in time:

- Linear time precomputation $O(|w|)$,
- Polylogarithmic access time $O(\log^2 |w|)$.

$O(\cdot)$ notation hides constants depending on $|A|$ but they are all polynomially bounded if A is deterministic and unambiguous.

Data structure idea

Counting reduction

Let $\tau = [\![A]\!](w)[k]$.

$\tau(x_1)$ is the first position p_1 for which
$\#[\![A]\!]_{x_1 \leq p_1}(w) \geq k$

	x_1	$\dots$	x_k
	$< p_1$	$\dots$	$\dots$
	p_1	$\dots$	$\dots$
	$\dots$	$\dots$	$\dots$
k	p_1	$\dots$	$\dots$
	$\dots$	$\dots$	$\dots$
	p_1	$\dots$	$\dots$
	$> p_1$	$\dots$	$\dots$

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- Binary search to find p_1: $O(\log n)$ calls to computing $\#[\![A]\!]_{x_1 \leq p}(w) \geq k$ by changing p
- We proceed recursively to find $\tau(x_2)$: first value p such that

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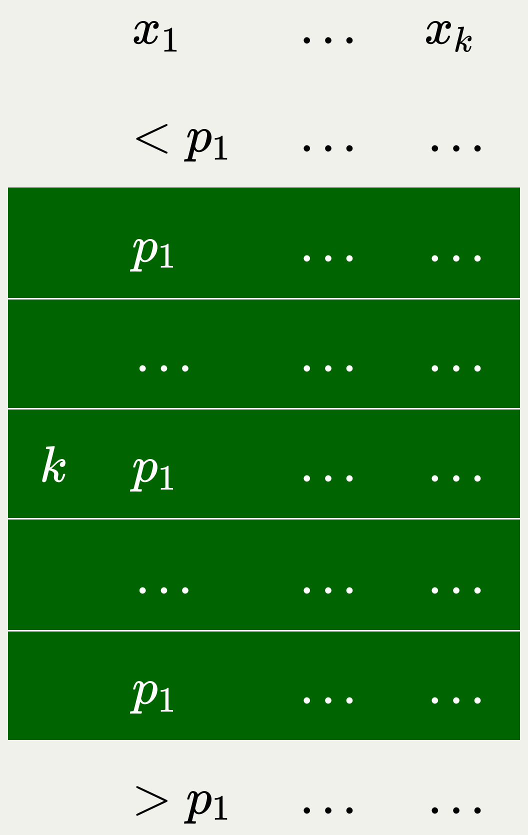

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	p_1	…	…
	…	…	…
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	…	…	…
	p_1	…	…
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$$\#[\![A]\!]_{x_1 = p_1, x_2 \leq p}(w) \geq k - \#[\![A]\!]_{x_1 < p_1}(w)$$

Maintaining matrix products

We can express $\#[\![A]\!]_{x \leq p}(w)$ as a matrix product (transition matrices):

$$P \cdot M_1 \dots M_n \cdot R$$

Moreover: $\#[\![A]\!]_{x \leq p}$ and $\#[\![A]\!]_{x \leq r}$ are the same product but for M_p and M_r.

Final data structure D: represents a matrix product $A_1 \cdot \dots \cdot A_r$ such that one can quickly update D so that it represents the product where A_i is replaced by B_i.

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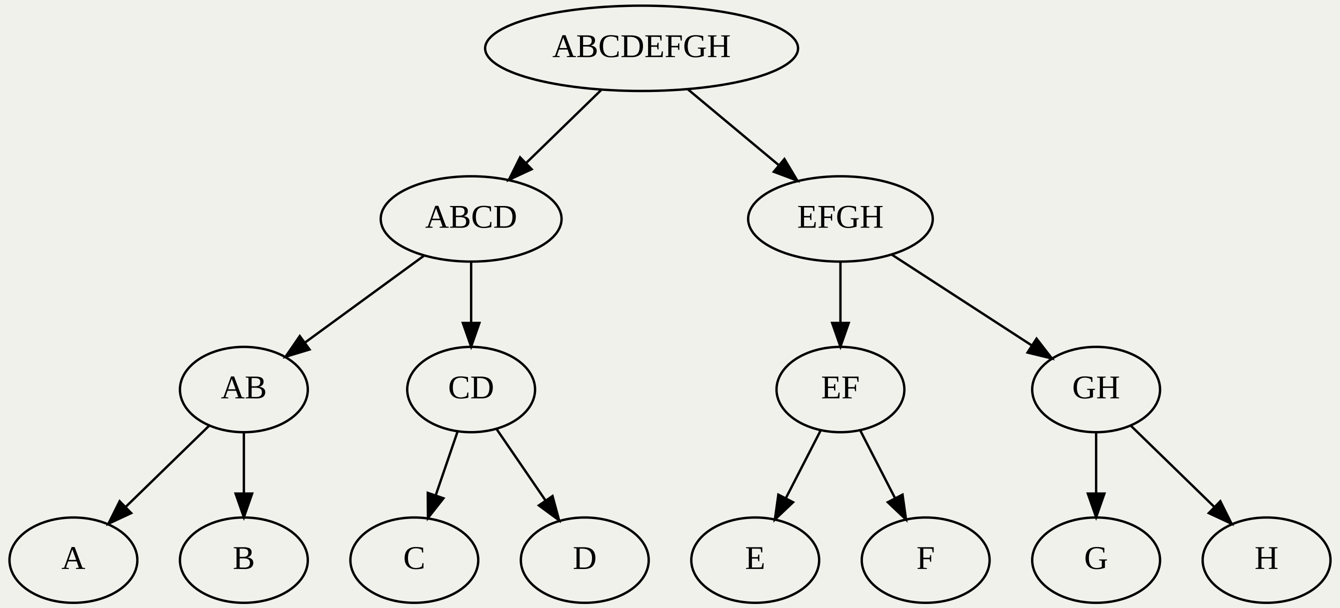

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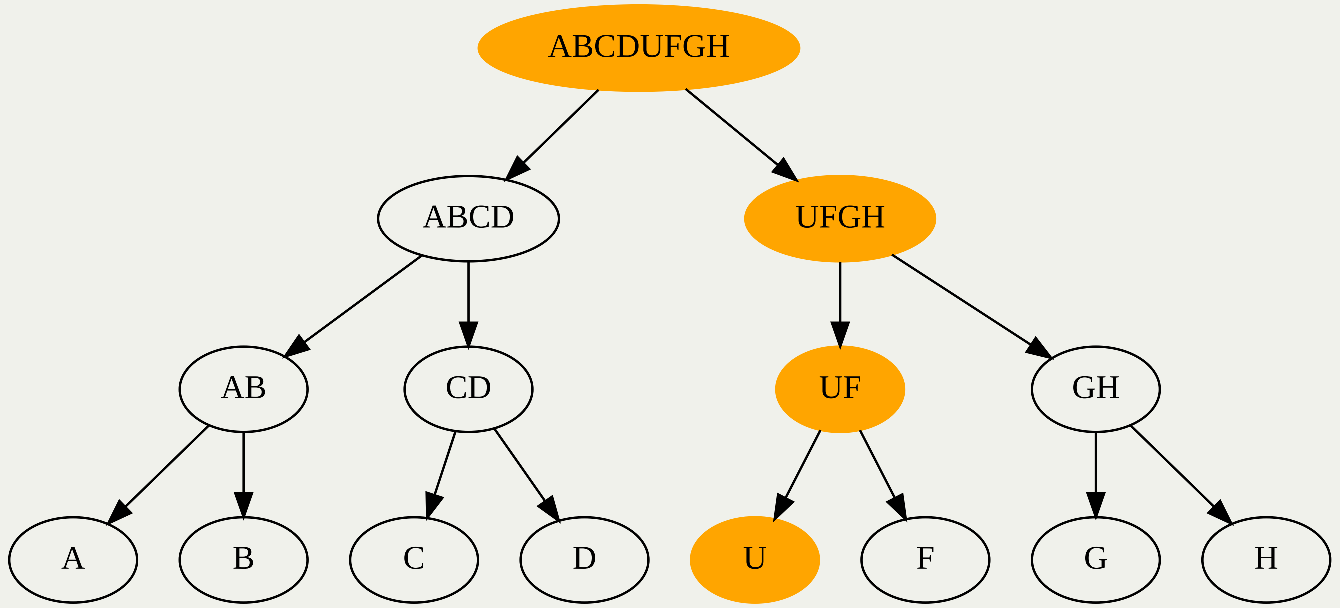

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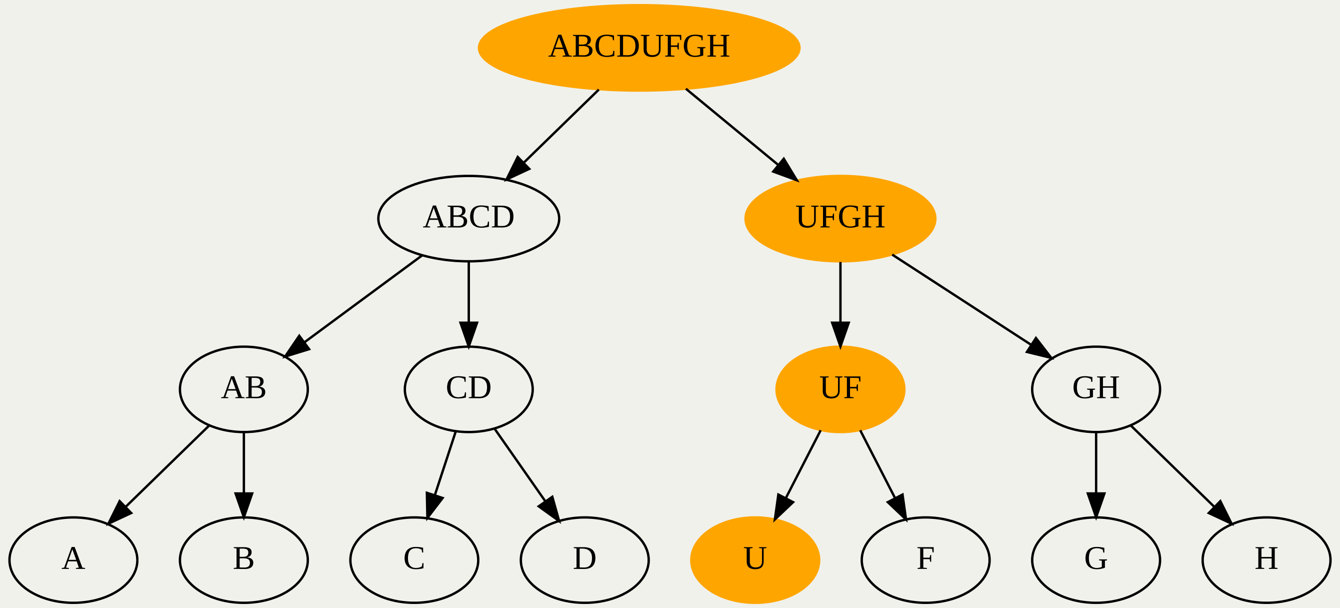

Update time: $O(\log n)$.

Dynamic words

Given data structures D_1 for w_1 and D_2 for w_2,
$k \leq |w_1|, i, j \leq |w_2|$:

$w_1 = a_1 \ldots a_n$, $w_2 = b_1 \ldots b_n$

Construct D_1' for w_1' and D_2' for w_2':

$w_1' = a_1 \ldots a_k \; b_i \ldots b_j \; a_{k+1} \ldots a_n$

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Possible $O(\log(n+m))$: use AVL tree operations to keep matrix product tree from balanced.

Future work

- Implementations:
    - reasonable data structures
    - updates may be cheap enough to maintain a query on a code base.
- Generalizations:
    - Orders that are not lexicographical
    - MSO over trees.